Information theory
PhD School on Quantum Field Theory, Strings and Gravity, Fall 2019

Solutions for problem set #n?

1. **A walkthrough on vacuum entanglement**

(a) Plugging in the ansatz $\Phi(t,x) = ce^{i(\pm kx \pm \omega t)}$ with uncorrelated signs, into the wave equation, we find

$$\left(-\partial_t^2 + \partial_x^2\right)\Phi = (k^2 - \omega^2)\Phi$$

meaning $\omega = |k|$ satisfies the wave equation.

(b) Plugging in the coordinate transformation

$$t = \frac{e^{a\xi}}{a}\sinh a\tau\ , \qquad x = \pm\frac{e^{a\xi}}{a}\cosh a\tau\ .$$

into the definition of the metric $ds^2 = -dt^2 + dx^2$ gives the Rindler metric

$$ds^2 = e^{2a\xi}\left(-d\tau^2 + d\xi^2\right)\ .$$

The equation satisfied by a massless scalar field in these coordinates is:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\Phi = e^{-2a\xi}\left(-\partial_\tau^2 + \partial_\xi^2\right)\Phi = 0$$

meaning, that, in complete anaglogy with the previous problem, the solution to the wave equation, in the right wedge is

$$\Phi^R(\tau,\xi) = \int_0^\infty \frac{ds}{\sqrt{4\pi s}}\left[b_s^R e^{is(\xi-\tau)} + b_s^{R\dagger}e^{-is(\xi-\tau)}\right] + (\tau\to-\tau,\ b_s^R \to b_{-s}^R)\ . \quad (1)$$

The left wedge is exactly the same, except that the Killing field ∂_τ is now past directed, so the creation and annihilation operators get swapped:

$$\Phi^L(\tau,\xi) = \int_0^\infty \frac{ds}{\sqrt{4\pi s}}\left[b_s^{L\dagger} e^{is(\xi-\tau)} + b_s^{L}e^{-is(\xi-\tau)}\right] + (\tau\to-\tau,\ b_s^L \to b_{-s}^L)\ . \quad (2)$$

The natural quantization condition is

$$\left[b_s^{R/L}, b_{s'}^{R/L\dagger}\right] = \delta(s-s')\ , \qquad \left[b_s^{L/R}, b_{s'}^{L/R}\right] = \left[b_s^{L/R\dagger}, b_{s'}^{L/R\dagger}\right] = 0$$

and the Rindler vacuum satisfies $b_s^{L/R}|0\rangle_R = 0\ \forall s$.

(c) In the original Minkowski coordinates, we can write the right moving sector of the fields as:

$$\Phi^R = \int_0^\infty \frac{ds}{\sqrt{4\pi s}}\left[b_s^R\{a(x-t)\}^{is/a} + b_s^{R\dagger}\{a(x-t)\}^{-is/a}\right]\ ,$$
$$\Phi^L = \int_0^\infty \frac{ds}{\sqrt{4\pi s}}\left[b_s^{L\dagger}\{a(t-x)\}^{is/a} + b_s^{L}\{a(t-x)\}^{-is/a}\right]\ .$$

To extract the annihilation operator, Notice that for $p>0$ we have the expression

$$\sqrt{\frac{p}{\pi}}\int dx\, e^{-ipx}\,\Phi(t=0,x) = a_p.$$

Now we write $\Phi(x) = \theta(x)\Phi^R(x) + \theta(-x)\Phi^L(x)$ and use the above expression. Carefully performing the Fourier transform (e.g. in `Mathematica`) gives

$$a_p = \int_0^\infty ds\, \frac{e^{\pi s/(2a)}}{2\pi\sqrt{ps}} \left\{ \left(\frac{a}{p}\right)^{-is/a} \Gamma\left(1 - \frac{is}{a}\right) \left(b_s^L - e^{-\pi s/a} b_s^{R\dagger}\right) - \left(\frac{a}{p}\right)^{is/a} \Gamma\left(1 + \frac{is}{a}\right) \left(b_s^R - e^{-\pi s/a} b_s^{L\dagger}\right) \right\} .$$

We conclude from this little exercise that

$$\left(b_s^R - e^{-\pi s/a} b_s^{L\dagger}\right)|0\rangle_M = \left(b_s^L - e^{-\pi s/a} b_s^{R\dagger}\right)|0\rangle_M = 0$$

for all s.

(d) For this one, we need the Baker-Campbell-Hausdorff formula

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \dots \tag{3}$$

We can start with

$$\begin{aligned}\left(b_s^R - e^{-\pi s/a} b_s^{L\dagger}\right)|0\rangle_M &= \left(b_s^R - e^{-\pi s/a} b_s^{L\dagger}\right) C e^{\left[-\int dk\, g(k)\, b_k^{L\dagger} b_k^{R\dagger}\right]} |0\rangle_R \\ &= C e^{\left[-\int dk\, g(k)\, b_k^{L\dagger} b_k^{R\dagger}\right]} \left(-e^{-\pi s/a} b_s^{L\dagger} + b_s^R + \int dk\, g(k) [b_k^{L\dagger} b_k^{R\dagger}, b_s^R] + \dots\right)|0\rangle_R \\ &= C e^{\left[-\int dk\, g(k)\, b_k^{L\dagger} b_k^{R\dagger}\right]} \left(-e^{-\pi s/a} - g(s)\right) b_s^{L\dagger}|0\rangle_R\end{aligned}$$

from which we conclude that

$$g(s) = -e^{-\pi s/a} .$$

For the Rindler observer, empty Minkowski space is a state populated with EPR pairs in the left and right wedges. Since the Rindler observer only has access to one side, the quantum state that she can access will be the reduced density matrix of the right wedge, which can be shown to be thermally populated.

2. **Hawking radiation generates too much entanglement**

In the previous problem we studied the entanglement properties of a horizon of infinite size. But the Schwarzschile black hole is finite and shrinking. At any given time, the EPR pairs produced will have wavelengths of order the horzion size, therefore, we need not consider a continuous integral over all wave modes, but it suffices to only include wave modes in a window around the horizon size. This gives us

$$|\Psi\rangle = |\psi_m\rangle \otimes e^{g a_k^{L\dagger} a_k^{R\dagger}} |0_{\rm Schw}\rangle$$

where ψ_m is the state of the quantum matter. Approximating this state by

$$|\Psi\rangle \approx |\psi_m\rangle \otimes \frac{1}{\sqrt{2}}\left(|0_L\rangle|0_R\rangle + |1_L\rangle|1_R\rangle\right)$$

and forming the reduced density matrix $\rho_R = \mathrm{Tr}_{M,L}|\Psi\rangle\langle\Psi|$ we find

$$\rho_R = \frac{1}{2}\left(|0_R\rangle\langle 0_R| + |1_R\rangle\langle 1_R|\right) .$$

which can be represented as $\rho_R = \frac{1}{2}\mathbb{1}_{2\times 2}$. The entanglement entropy $S_{\rm ent} = -\mathrm{Tr}_R\, \rho_R \log \rho_R = \log 2$

3. At step 2, the reduced density matrix on the right is

$$\rho_R = \frac{1}{4}\left(|0_{R_1}\rangle\langle 0_{R_1}| + |1_{R_1}\rangle\langle 1_{R_1}|\right) \otimes \left(|0_{R_2}\rangle\langle 0_{R_2}| + |1_{R_2}\rangle\langle 1_{R_2}|\right) \ .$$

   which can be represented as $\rho_R = \frac{1}{4}\mathbb{1}_{4\times 4}$. Computing the entanglement entropy of this reduced density matrix yields $S_{\text{ent}} = 2\log 2$. It is quite easy to guess that at the N^{th} step the entanglement entropy will be $S_{\text{ent}} = N\log 2$.

4. After N such steps if the black hole is completely evaporated, the final state of the radiation should be pure, as there is nothing left for it to be entangled with. But we have shown that the entanglement entropy of the radiation is $N\log 2$. It seems we have evolved from a pure state to a mixed state. But this can't be. We have derived this in an absurdly simple aproximation scheme! As you will see in the lecture, if we assume that physics is local and without drama at the horizon and we pick our spatial slices to be weakly curved everywhere (avoiding the singularity), then *we can not avoid this conclusion.* One of our assumptions must break down, e.g. that physics is not local, or that there is drama at the horizon. The point of the information paradox is to understand *how* these assumptions break down.

5. Good luck. The future rests in your hands.