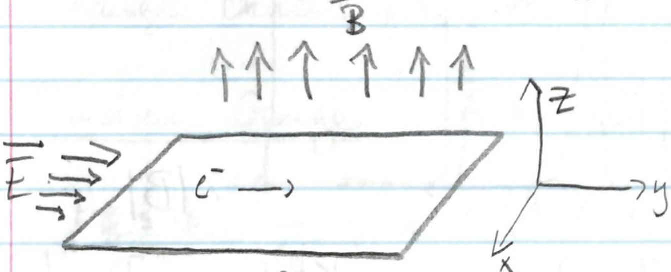
The Quantum Hall Effect
Explained Simply by a Simple Person (bis)
MIT Grad Student Seminar
April 22nd, 2016

Main reference: David Tong: "The Quantum Hall Effect"
→ See references therein but good ones include
Girvin: QHE
Jain: Composite Fermions
Fradkin: Field theories of Condensed Matter Physics
Wen: . . . .
Witten: 3 lectures on topological Phases of Matter

Basics: (Classical Hall Effect) 1879, Edwin Hall



0 in steady state

Force on single electron: $m\frac{d\vec{v}}{dt} = -e\left[\vec{E} + \vec{v}\times\vec{B}\right] - \frac{m\vec{v}}{\tau}$ → scattering time

Ohm's law: $\vec{J} = \overleftrightarrow{\sigma}\cdot\vec{E}$ or $\vec{E} = \overleftrightarrow{\rho}\cdot\vec{J}$
conductivity / resistivity Tensor

$\vec{J} = -\frac{N}{A}e\vec{v} = -ne\vec{v}$

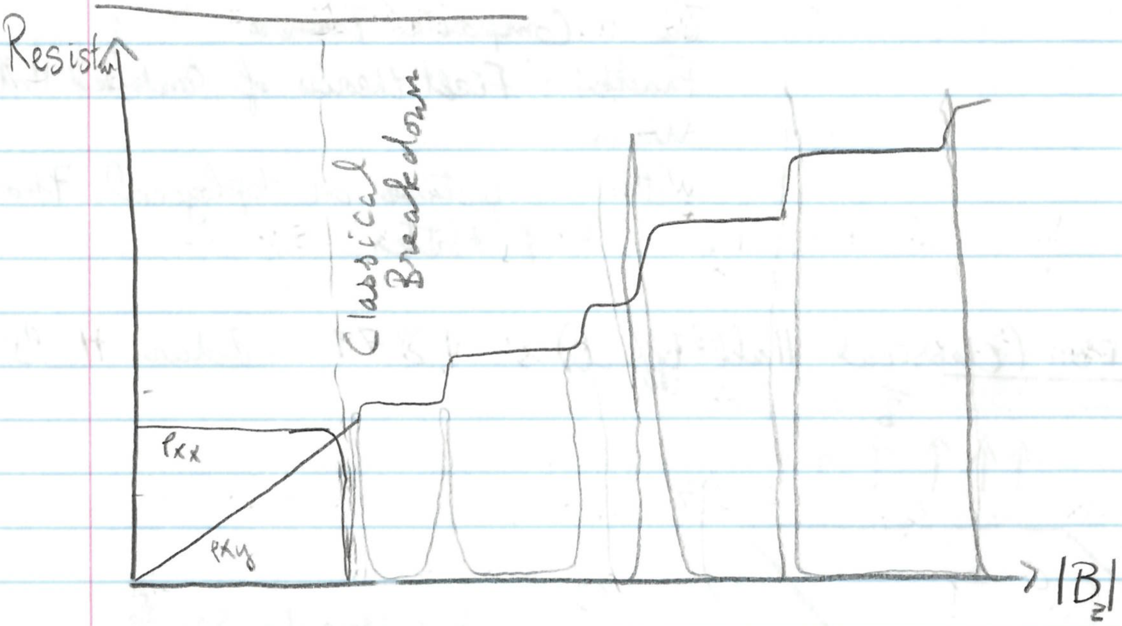
Find: $\rho = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix}$ w/ $\rho_{xx} = m/ne^2\tau$

$\rho_{xy} = \frac{\cancel{e}|B|}{\cancel{m}}\cancel{\tau}\frac{\cancel{m}}{ne^{\cancel{2}}\cancel{\tau}} = \boxed{\frac{|B|}{ne}}$

Nice properties: ρ_{xy} is Independent of τ and since we usually measure resistance (rather than resistivities), ρ_{xy} is nice since it is independent of geometric factors

$$R_{xy} = \frac{V_y}{I_x} = \frac{\cancel{L_y} E_y}{\cancel{L_y} J_x} = -\rho_{xy} \checkmark \qquad R = \left(\frac{L}{A}\right)\rho$$

Classical Prediction:



Experiment from 1980 by von Klitzing Dorda & Pepper

Find $\rho_{xy} = \frac{2\pi\hbar}{e^2} \times \frac{1}{\nu}$ for $\nu \in \mathbb{Z}$

$\rho_{xx} = 0 \rightsquigarrow \sigma_{xx} = \frac{\rho_{xx}}{\rho_{xx}^2 + \rho_{xy}^2} = 0 \rightsquigarrow$ Insulator

In 1982 Stui Stormer & Gossard observe plateaus near $\nu = \frac{1}{3}$ & $\frac{2}{3}$ (Later) $\frac{1}{5}, \frac{2}{5}, \frac{3}{7} \ldots$ (Lowest LL)

$\nu = \frac{4}{3}, \frac{5}{3}, \frac{7}{5}, \frac{5}{2} \ldots$

80 fractional plateaus seen to date!

→ Find Quantum effect in macroscopic system!
→ Implies the behavior must be "robust" to many body / dirty physics arising in large samples

→ Phase transition does not have ~~an~~ local order parameter
→ New type of transition / way of characterizing states

<u>Low brow Approach</u> to <u>IQHE/FQHE</u>: Electron wavefunction

⇝ In high $\vec{B}$ field can neglect electron spin (will be pointing in $\vec{B}$-field direction), treat electrons as non-interacting

$$H = \frac{1}{2m}\left(\vec{P} + e\vec{A}\right)^2$$ ⇝ for a single electron

Wavefunctions highly degenerate and depend on gauge choice for $\vec{A}$

<u>Landau Gauge</u>: $\vec{A} = x|B|\hat{y}$, $\psi_k(x,y) = e^{iky} f_k(x)$

$$H_k \psi_k = \frac{1}{2m}\left(P_x^2 + (\hbar k + e|B|x)^2\right)\psi_k$$ ⇝ Shifted H.O.

$$H = \frac{1}{2m}P_x^2 + \frac{m\omega_B^2}{2}\left(x + k l_B^2\right)^2 \qquad \omega_B = \frac{e|B|}{m} \qquad l_B = \sqrt{\frac{\hbar}{e|B|}}$$

$$\psi_{n,k} = e^{iky} H_n(x + k l_B^2)\, e^{-(x + k l_B^2)^2 / 2 l_B^2}$$

$$E_n = \hbar\omega_B\left(n + \tfrac{1}{2}\right)$$ <u>indep of k</u>

Degeneracy of Landau level: $k_{max} = \frac{2\pi N}{L_y}$

k_{max} needs to lie with sample: $k_{max} l_B^2 = L_x$

$$\therefore \boxed{N = \frac{L_y L_x}{2\pi l_B^2} = \frac{e|B|A}{2\pi\hbar}}$$

Add electric field: $H \to H - e|E|x$

$$\psi(x,y) = \psi_{n,k}\left(x - m|E|/e|B|^2, y\right)$$

$$E_{n,k} = \hbar\omega_B\left(n+\tfrac{1}{2}\right) + eE\left(k l_B^2 - \frac{eE}{m\omega_B^2}\right) + \frac{m}{2} E^2/B^2$$

Symmetric Gauge (for later)

$$A = -\frac{y}{2}|B|\hat{x} + \frac{x}{2}|B|\hat{y}$$

Define $z = x - iy$, $\bar{z} = x + iy$

$$a = i\sqrt{2}\left(l_B \partial_{\bar{z}} - \frac{z}{4l_B}\right), \quad a^\dagger = -i\sqrt{2}\left(l_B \partial_z - \frac{\bar{z}}{4l_B}\right)$$

$$b = -i\sqrt{2}\left(l_B \partial_z + \frac{\bar{z}}{4l_B}\right), \quad b^\dagger = i\sqrt{2}\left(l_B \partial_{\bar{z}} + \frac{z}{4l_B}\right)$$

$$|n,m\rangle = \frac{a^{\dagger n} b^{\dagger m}}{\sqrt{n!m!}}|0,0\rangle, \quad a|0,0\rangle = b|0,0\rangle = 0$$

$$|0,0\rangle \sim e^{-|z|^2/4l_B^2}$$

$$E_n = \hbar\omega_B\left(n + \frac{1}{2}\right)$$

$[a,a^\dagger] = [b,b^\dagger] = 1$, $H = \hbar\omega_B(a^\dagger a + \frac{1}{2})$ $\quad |0,m\rangle = \left(\frac{z}{l_B}\right)^m e^{-|z|^2/4l_B}$

Quick & Dirty derivation of IQH conductivities

Given that $m\dot{\vec{x}} = \vec{p} + e\vec{A}$

$$\vec{I} = -\frac{e}{m}\sum_{\text{filled states}} \langle\psi| -i\hbar\vec{\nabla} + e\vec{A}|\psi\rangle$$

$\therefore \quad I_x = 0$

$$I_y = -\frac{e}{m} \sum_{n=1}^{\nu} \sum_{k} \langle \psi_{n,k} | -i\hbar\partial_y + e|B|x | \psi_{n,k} \rangle$$

$$= -\frac{e}{m} \sum_{n=1}^{\nu} \sum_{k} \hbar k + e|B| \underbrace{\langle x \rangle_{nk}}_{-\frac{\hbar k}{e|B|} + mE/e|B|^2}$$

$$AJ_y = I_y = -e\nu \sum_k \frac{E}{B} = -e\nu \left(\frac{e|B|A}{2\pi\hbar} \right) \frac{E_x}{|B|}$$

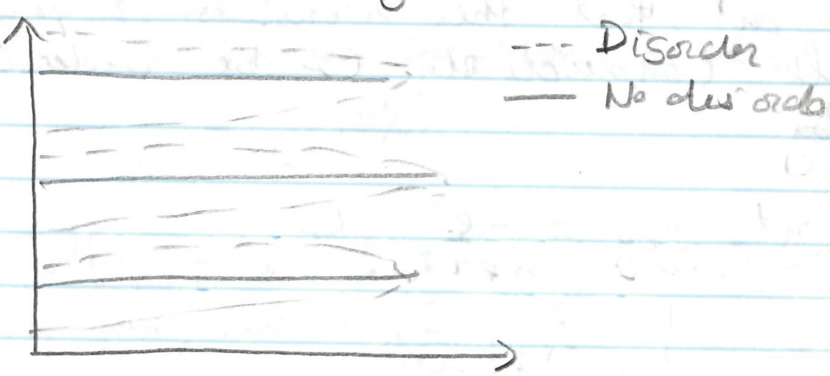
$$\boxed{\frac{E_x}{J_y} = -\rho_{xy} = -\frac{2\pi\hbar}{e^2\nu}, \quad \rho_{xx} = 0}$$

We find that ν in the hall resistivity is precisely the number of filled Landau levels

→ This was too quick, it does not explain the existence of platteaux, nor why there are jumps between them

Disorder: Samples are dirty (including ones where Hall resistivities were measured)

Effect (1) degeneracy lifted

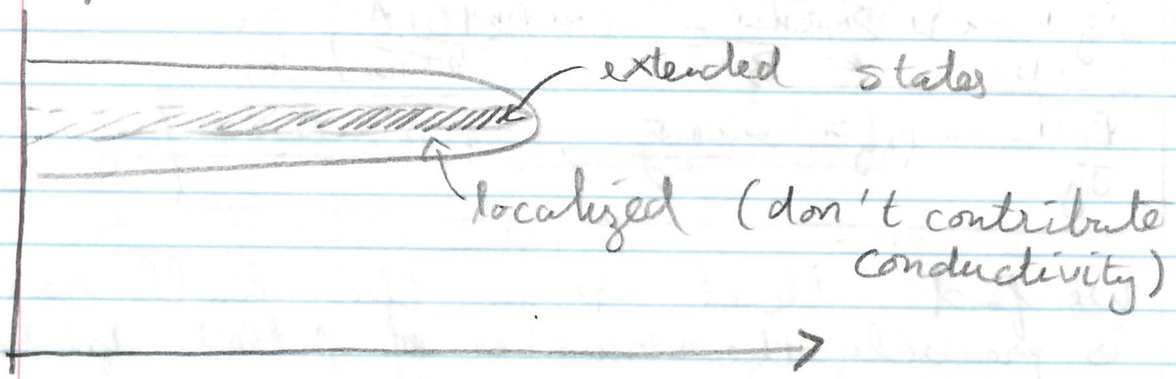


Effect (2) states become localized (generic effect of weak disorder, e.g. ~~A~~Anderson)

$\partial_t \langle \vec{X} \rangle \perp \vec{\Delta} V$, for a random potential

This means particles centered near extrema are localized

↳ These are the states at The Edges of bands



→ If we've filled the extended states in some LL and decrease $|B|$, we start to populate localized states rather than jump down to next LL but localized states to contribute to conductance → explains plateau!

⇝ Extended states live on the edge

<u>Topology (brief)</u>

It turns out that the robustness & quantization of the Hall Conductivity can be understood using Topolog

It turns out $\sigma_{xy} = \frac{-e^2}{2\pi\hbar} C$

where $C = \int_{\square} F_{xy} \equiv$ Chern number $\in \mathbb{Z}$

↑ Field strength associated to Berry's connection as parameters in H are varied

σ_{xy} is a topological invariant of the System

This is called the Thouless, Kohomoto, Nightingale, Nij formula (read about it!)

Low brow Approach to FQHE

Now we need to understand non-integer plateaus. These arise due to Coulomb repulsion between electrons

$$V = \sum_{i<j} \frac{e^2}{|\vec{r}_i - \vec{r}_j|}$$

Let us take $\nu < 1$ so first LL is partially filled.

There are still a macroscopic # of degenerate states that will be lifted due to V

But its hard to solve this in perturbation theory.

Laughlin guessed the answer.

Recall single electron states in LLL were

$$z^m e^{-|z|^2/4l_B}$$

Fully filled LL has a Slater determinant structure

$$\begin{vmatrix} z_1^0 & z_2^0 & \cdots & z_N^0 \\ z_1 & z_2 & \cdots & z_N \\ \vdots & & & \\ z_1^{N-1} & & & z_N^{N-1} \end{vmatrix} \times e^{-|z|^2} = \prod_{i<j} (z_i - z_j)\, e^{-|z|^2/4l_B^2}$$

Laughlin guesses: $\prod_{i<j} (z_i - z_j)^m$

For the $\nu = \frac{1}{m}$ plateau.

To do this one repeats a version of the calculation before and finds $\nu = \frac{1}{m}$

⇝ Laughlin Wavefunctions have many interesting properties : quasi-hole excitations with e/m fractional charge & anyonic statistics
⇝ used to get other than $1/m$ filling

→ Also generalized : (see Read + Moore
or Read + Rezayi)

⇝ Don't have time to discuss

<u>High brow approach to IQHE/FQHE : Chern-Simons</u>

Recall that we said that the conducting states in the sample live on the edge due to disorder. This idea gives us an extra handle.

In QFT, want to compute: $Z[A_\mu] = \int D[\phi,\psi]\, e^{iS[A,\phi,\psi]}$

$$= e^{iS_{eff}[A_\mu]/\hbar}$$

$$\frac{\delta \log Z}{\delta A_\mu} = \langle J^\mu(x) \rangle$$ ⇝ encodes response to external fields.

Hard to compute Seff in general, but we know two things: 1) the system is gapped, so Seff is local

2) Seff should be gauge invariant (actually Z)

does not contribute to hall conductance

$$S_{eff} = S^{4d}_{maxwell}[A] + S^{3d}_{CS}[A] + \ldots$$

$$S_{CS}[A] = \frac{k}{4\pi}\int d^3x\, \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

$$J_i = \frac{\delta S_{CS}}{\delta A_i} = -\frac{k}{2\pi}\epsilon_{ij} E_i$$

$$J_0 \equiv \text{charge density} = \frac{\delta S}{\delta A_0} = \frac{k}{2\pi}|B| = \frac{k}{2\pi}F_{12}$$

$\sigma_{xy} = \frac{k}{2\pi}$ ⇝ to match hall conductance $k = e^2\nu/\hbar$ ⇝ quantized

But k is quantized because S_{CS} is only gauge invariant up to a total derivative, which contributes on spaces with topology ⇝ forces k to be quantized as above (also comes from dirac quantization)

What about FQHE?

Say we computed $Z[A_\mu]$ in two steps, e.g. there is an emergent dynamical $U(1)$ gauge field that arises @ low energies

$$Z[A_\mu] = \int \mathcal{D}a_\mu\, e^{iS[a,A]/\hbar}$$

The only conserved current built out of a_μ th A_μ can couple to is

$$J^\mu = \frac{e^2}{2\pi\hbar}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho$$

makes sure minimum charge of $J = e$

Postulate: $S = \frac{e^2}{\hbar}\int d^3x\, \frac{1}{2\pi}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu a_\rho - \frac{m}{4\pi}\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$

Can integrate out $a_\mu = \frac{1}{m} A_\mu$ and get

$S^{\text{eff}}_{\text{eff}} = \frac{e^2}{4\pi m} \int \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho \rightsquigarrow \sigma_{xy} = \frac{e^2}{2\pi\hbar}\left(\frac{1}{m}\right)$

$\rightsquigarrow$ as expected in the Laughlin state.

→ This seems to contradict quantization of the level

$\rightsquigarrow$ have to be more careful in integrating out in cases where this matters. (keep a_μ)

Fractional Charge + Statistics

Let's turn off A and couple a to a charged particle at the origin $j^0 = e\,\delta(x)$ ↳ related

EOM: $\frac{e^2}{2\pi\hbar} f_{\mu\nu} = \frac{1}{m} \epsilon_{\mu\nu\rho} j^\rho \to \frac{1}{2\pi} f_{12} = \frac{\hbar}{em} \delta^2(x)$

Particle of charge e also has a magnetic flux $\hbar/em$

Now if we recouple A: find $J^0 = \frac{e^2}{2\pi\hbar} f_{12} = \frac{e}{m}\delta^2(x)$

$\rightsquigarrow$ Need to have m of these for Dirac quant on S^2

→ Fractional Statistics can be read off from the fact that exchange of such particles with flux lead to Aharanov Bohm phase

$\Phi = \frac{2\pi\hbar}{em} \rightsquigarrow$ excitations are anyonic

⇒ All of these features appear in the Laughlin wavefunctions & generalizations.

→ Relation: (by now you're all tired so I'll be brief)

It turns out that Chern-Simons theory with a boundary has a description as a chiral boson living on the boundary

It turns out this chiral boson theory is a CFT and correlation functions reproduce Laughlin wavefunction + Moore Read + . .

= + Majorana