Interesting Saddles
Lecture 3
Large-c CFT

1) Review of 2d CFT

As with the overall theme of these lectures, we increase by 1 dimension today. In 2d many simplifications happen if we have Lorentz symmetry & local conformal invariance; which we now review.

diffeomorphism acts on metric $\eta_{\mu\nu}$ as.

$$ds^2 \to ds^2 + (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) dx^\mu dx^\nu$$

So if we want ϵ to be a conformal transformation $(ds^2 \to \Omega(x) ds^2)$

$$\boxed{(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = \# \eta_{\mu\nu} \quad (*) \qquad \text{where } \# = \frac{2}{d} \partial_\mu \epsilon^\mu}$$

by contracting with η

In $d=2$ in Euclidean signature; $(*)$ is the Cauchy-Riemann condition

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2, \quad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1$$

$\therefore$ Conformal transformations in 2d coincide with holomorphic maps:

$$x + iy \equiv z \to f(z), \quad x - iy \equiv \bar{z} \to \bar{f}(\bar{z})$$

$$\therefore dz d\bar{z} \to \left| \frac{\partial f}{\partial z} \right|^2 dz d\bar{z}$$

$\rightsquigarrow$ Physics must be "invariant" under these conformal maps. $\rightsquigarrow$ There is a subtlety to this, which I will get to momentarily and explains the quotation marks under such a diffeo the action transforms

$$\delta S = \int T^{\mu\nu} \delta g_{\mu\nu} = \int T^{\mu\nu} (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$$

$$= \int T^{\mu\nu} \eta_{\mu\nu} \left(\frac{2}{d} \partial_\rho \epsilon^\rho\right)$$

$\rightsquigarrow$ Invariance implies tracelessness of $T^{\mu\nu}$

In holomorphic coordinates: $T^\mu{}_\mu = T_{z\bar{z}} = T_{\bar{z}z} = 0$

and we define: $T(z) \equiv T_{zz}$, $\bar{T}(\bar{z}) = T_{\bar{z}\bar{z}}$

which follows from $\boxed{\partial_\mu T^{\mu\nu} = 0 \ \& \ T^\mu_\mu = 0}$

Ward Identities

As a result of the symmetries of 2d can derive the following:

$$\boxed{\langle T(z) \phi_1(w_1, \bar{w}_1) \dots \phi_n(w_n, \bar{w}_n) \rangle = \sum_{i=1}^{n} \left(\frac{h_i}{(z-w_i)^2} - \frac{\partial_{w_i}}{(z-w_i)} \right) \times \langle \phi_1(w_1, \bar{w}_1) \dots \phi_n(w_n, \bar{w}_n) \rangle}$$

h_i are known as conformal dimensions of primary fields

under $z \to f(z)$

$$\phi_i(z, \bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^{h_i} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}_i} \phi(f, \bar{f})$$

$T(z)$ almost transforms like a primary of dimension 2

Central Charge under $z \to f(z)$

$$T(z) \to (\partial f)^2\, T(f(z)) + \frac{c}{12}\{f(z), z\}$$

$$\boxed{\{f(z),z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 \equiv \text{Schwarzian derivative}}$$

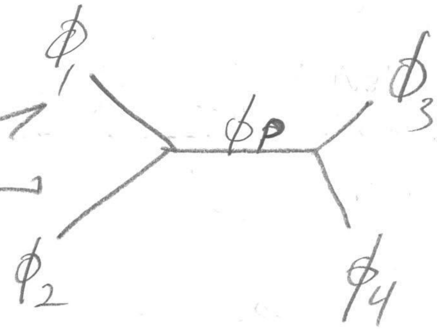
c is the central charge, & we will see that cool stuff can happen when c gets big.

<u>Why</u>? We will try & see how to combine the anomalous transformation of $T(z)$ with the conformal ward identity.

Now I promised some Saddles, but saddles of what? And <u>why</u>? In statistical mechanics we want Saddles of Z as these describe different phases of our system as a function of β.

In CFT we are interested in correlation functions of primary operators:

$$\langle \phi_1(z_1)\,\phi_2(z_2)\,\phi_3(z_3)\,\phi_4(z_4)\rangle = \sum_p$$

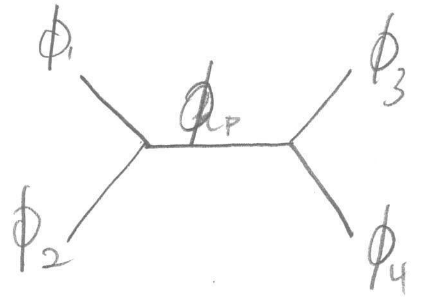


$\rightsquigarrow$ If somehow each  $\sim e^{\#}$

then we could use the saddle point approximation

Zamolodchikov '86 showed that @ $c \to \infty$, $\frac{h_i}{c}$ fixed

then



$$= e^{-c/6\, f_p\left(\frac{h_i}{c}, z\right)} \equiv \mathcal{F}$$

Conf. block

$$z = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

& f_p is a semiclassical block

Also if $\phi_1 = \phi_2$ & $\phi_3 = \phi_4$, ϕ_p can be $\mathbb{1}$

Now I haven't specified a particular theory of interest, meaning we don't know the possible ϕ_p that can propapapagate, but we <u>do</u> know that, as $z_1 \to z_2$, $\mathcal{F} = (z_1 - z_2)^{-2h + h_p}$, So $h_p = 0$ is the biggest in this limit

The magic of large c CFT is in assuming the <u>whole</u> sum localizes on $h_p = \mathbb{1}$, the saddle.

<u>Why is $h_p = \mathbb{1}$ interesting</u>? Isn't this just the disconnected part of the correlation function? No!

Because 2d CFT is governed by Conformal symmetry

$\mathcal{F}$ captures the exchange of <u>$\phi_p = \mathbb{1}$ and all descendants</u>

but $T(z) = L_{-2}\mathbb{1}$ so the stress tensor is not a primary, which we already deduced by how it transforms.

$\rightarrow$ This block captures stress tensor exchanges

Example: Heavy-heavy, light-light block.
HHLL

Say we are interested in $\langle \phi_H(\infty)\, \phi_L(1)\, \phi_L(x)\, \phi_H(0) \rangle$
with $H/c \gg \frac{h_L}{c}$ $\qquad \simeq e^{-c/6\, f_{\mathbb{1}}}$

By the ward Identity

$$\frac{\langle T(z)\, \phi_H(0)\, \phi_L(1)\, \phi_L(x)\, \phi_H(\infty) \rangle}{\langle \phi_H\, \phi_L\, \phi_L\, \phi_H \rangle} = \frac{H}{z^2} + \frac{h_L}{(z-x)^2} + \frac{h_L}{(z-1)^2} + \frac{2h_L}{z(1-z)} - \frac{c}{6}\frac{\partial f}{\partial x}\, \frac{x(1-x)}{z(z-x)(1-z)}$$

Now recall $H \gg h_L$ $\quad$ So $\simeq \frac{H}{z^2}$

As an approximation to the whole correlation fct. we can go to a coordinate $w(z)$ where

$$(w'(z))^2 \frac{H}{w(z)^2} + \frac{c}{12}\{w(z), z\} = 0 \longrightarrow \text{called a uniformization}$$

Solved by: $w = z^{\sqrt{1-\frac{24H}{c}}}$ then the 4pt function is approximated by 2pt function $\langle \phi_L \phi_L \rangle$ in the w coordinate

$$\boxed{(w'(1))^{h_L}(w'(x))^{h_L}\frac{1}{(w(1)-w(x))^{2h_L}}=\left(\frac{x^{\frac{1-\alpha}{2}}(1-x)^{\alpha}}{\alpha(1-x)}\right)^{-2h_L} \qquad \alpha=\sqrt{1-\frac{24H}{c}}}$$

What's so interesting about this? Let's put $x=e^{i\theta}$

$$\rightarrow \langle \phi_H \phi_L \phi_L \phi_H \rangle = \left[\frac{2}{\alpha}\sin\left(\frac{\alpha\theta}{2}\right)\right]^{-2h_L}$$

$$\alpha=\sqrt{1-24H/c} \qquad \text{if } H>\frac{c}{24}$$

$$\sin \rightarrow \sinh$$

This looks like a correlator in a thermal state. $H=c/24$ is actually the BTZ threshold in AdS_3. There are no BH solutions below this value.

$\rightsquigarrow$ we've formed a BH from a pure state!

In this language, solving the info paradox is equivalent to restoring non-thermal physics by including other exchanges.

Many more things to do with this, if you're interested, come talk to me!