Sources: 1) Mezard + Parisi + Virasoro
Spin Glass theory & beyond
2) Castellani & Cavagna
Spin Glass theory for pedestrians
3) Denef TAST lectures

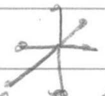
Saddle point Methods
& Their uses in Spin Glasses
Lecture 1: April 22nd

The point of these lectures will be that large-N + a path integral can result in some very beautiful physics, applicable in a range of physical situations, from stat mech → BHs

Roughly speaking, the lectures will increase in dimension by one increment each time. Today, we will be in 0+0 dimensions.

Spin Glasses a) Recall the Ising model

$$H = -J\sum_{\langle ij\rangle} \sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma^i \qquad \sigma^i = \pm 1$$

In general for $\langle ij\rangle \Rightarrow$  this is not solvable exactly
@ large coordination number & large N, we can get an approx solution

Expand around the "minimum" $\{\sigma_i\} = \{m_i + \delta\sigma_i\}$

$$H \approx -J\sum_{\langle ij\rangle} m_i m_j + m_i\delta\sigma_j + m_j\delta\sigma_i - h\sum_i \sigma_i$$

By translation inv. $m_i = m \Rightarrow$

$$H^{MF} = -J\sum_{\langle ij\rangle} m^2 + m(\sigma_i + \sigma_j - 2m) - h\sum_{i=1}^{N}\sigma_i$$

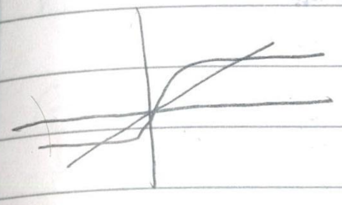
$$H^{MF} = -Jm\sum_{\langle ij\rangle}(\underbrace{\sigma_i + \sigma_j}_{2\sigma_i} - m) - h\sum_{i=1}^{N}\sigma_i$$

$$= -Jm\frac{1}{2}\sum_i \underbrace{\sum_{j\,\mathrm{nn}\,i}}_{q}(2\sigma^i - m) - h\,\sigma^i$$

$$Z = e^{-\beta H} \approx \sum_{\sigma_i = \pm 1} e^{-\beta\left[\frac{JmNq}{2} - (Jmq+h)\sum_i \sigma_i\right]}$$

$$= e^{-\beta J m N q/2} \left\{ 2\cosh\left[\beta(h + Jmq)\right]\right\}^N$$

Now $m = \frac{1}{N}\frac{\partial}{\beta \partial h}\log Z = \tanh(\beta(h + Jmq))$



$\rightsquigarrow$ predicts a phase transition

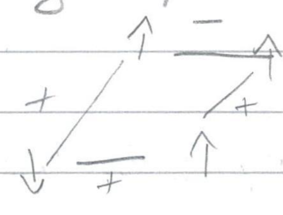
$$\boxed{\frac{1}{\beta_c} = Jq} \qquad m = 0 \text{ vs } m \neq 0$$

$\rightsquigarrow$ This solution required: $\rightsquigarrow$ fluctuations small ($N \gg 1$)
$\rightsquigarrow$ Expanding around a saddle

Feature: Critical temperature $\Rightarrow$ Exchange of saddle dominance.

$\rightarrow$ The features of this model are somewhat boring so can we spice it up, in particular the saddles describe a paramagnetic $\Rightarrow$ ferromagnetic transition.

One way to spice things up is to imagine describing a <u>dirty</u> system



$\rightsquigarrow$ can not satisfy all bonds

<u>P-spin model</u> $\quad H = -\sum_{i_1 \ldots i_p}^{N} J_{i_1 \ldots i_p} \sigma_{i_1} \ldots \sigma_{i_p} \qquad p \geq 3$

subject to $\sum_i \sigma_i^2 = N$

The couplings are <u>random</u> $\pm$: $P(J) = \exp\left\{-\frac{J_{i_1 \ldots i_p}^2}{2}\frac{2N^{p-1}}{p!}\right\}$

Now we want to compute the free energy

$$\boxed{\overline{F} = \int dJ\, P(J) \log Z}$$

but Z is impossible to compute for any particular J_{ij}...

$\rightsquigarrow$ Replica trick $\log Z = \lim_{n\to 0} \frac{Z^n - 1}{n} = \partial_n Z^n \Big|_{n=0}$

We will compute $\overline{Z^n}$ for n integer & take $n \to 0$ at the end

For $n \in \mathbb{Z}$: $\overline{Z^n} = \int D\sigma_i^a \prod_{i_1 \ldots i_p} e^{-J_{i_1\ldots i_p}^2 \frac{N^p}{p!} + \beta J_{i_1 \ldots i_p} \sum_{a=1}^n \sigma_{i_1}^a \ldots \sigma_{i_p}^a}$

Complete square

$$= \int D\sigma_i^a \exp\left[\frac{\beta^2}{4N^{p-1}} \sum_{a,b}^n \left(\sum_{i=1}^N \sigma_i^a \sigma_i^b\right)^p\right.$$

Now let us introduce an orderparameter, the overlap matrix

$$Q_{ab} = \frac{1}{N} \sum_i \sigma_i^a \sigma_i^b \,, \quad 1 = \int dQ_{ab}\, \delta(NQ_{ab} - \sum_i \sigma_i^a \sigma_i^b)$$

$$= \int DQ_{ab}\, D\lambda_{ab}\, e^{i\lambda_{ab}(NQ_{ab} - \sum_i \sigma_i^a \sigma_i^b)}$$

$$\therefore \overline{Z^n} = \int DQ\, D\lambda\, D\sigma \exp\left\{ \frac{\beta^2}{4} N \sum_{ab} Q_{ab}^p + N\tilde{\lambda}_{ab} Q_{ab} - \tilde{\lambda}_{ab} \sum_i \sigma_i^a \sigma_i^b \right.$$

$$= \int DQ \exp\left\{ \frac{\beta^2}{4} N\, Q_{ab}^p + N\lambda_{ab} Q_{ab} - \frac{N}{2} \log\det 2\lambda_{ab} \right\}$$

large N, can evaluate by saddle point $2\lambda_{ab} = Q_{ab}^{-1}$

$$\overline{Z^n} = \int DQ \exp\left\{ \frac{N}{2}\left[\frac{\beta^2}{2} Q_{ab}^p + \log\det Q_{ab} \right]\right]$$

Saddle point equations: $\frac{\beta^2}{2} p \, Q_{ab}^{p-1} + (Q^{-1})_{ab} = 0$

⤳ What is Q_{ab}? $\frac{1}{N}\vec{\sigma}^a \cdot \vec{\sigma}^b$ ⤳ It computes how aligned two different realizations of the system is.

Guess $Q_{ab} = \begin{pmatrix} 1 & & q \\ & 1 & \\ q & & 1 \end{pmatrix}$ where q is the order parameter $q < 1$

⤳ This is known as the replica symmetric Ansatz

⤳ $n = 0$: $\frac{\beta^2 p}{2} q^{p-1} + \frac{q}{(1-q)^2} = 0$ $q = 0$ is always a solution

Also: $q^{p-2}(1-q)^2 = \frac{2}{p} T^2$ ⤳ if $T \gg 1$ no solution since $q < 1$

but below T_{crit}

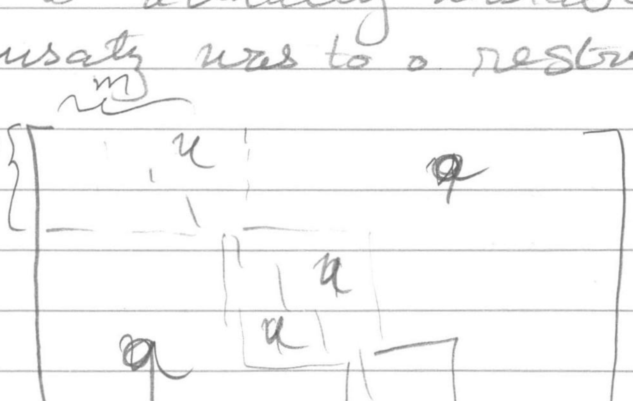
If we compute

$S = \partial_{\beta^2} F < 0$ at low T. This makes little sense

Also, this solution is actually unstable.

→ Our ansatz was too restrictive

1step RSB $Q_{ab} =$



$$F_{1RSB} = -\frac{1}{2\beta}\left\{ \frac{\beta^2}{2}\left(1 + (m-1)u^p - mq^p\right) + \left\{1 - \frac{1}{m}\right\}\log(1-u) + \frac{1}{m}\log\left[m(u-q) + 1-u\right] + \frac{q}{m(u-q)+1-u} \right\}$$

Need to minimise F w.r.t m, q, u

→ $q=0$ & $m=1$ & $u \neq 0$ below a critical temp

→ This T_c is a glass transition temp

→ This solution is the correct low energy minimum and suggests the Free energy landscape is broken up into <u>many</u> equilibrium states

→ These are characterized by their interesting overlaps.

→ Spin glass transition