Information theory

PhD School on Quantum Field Theory, Strings and Gravity, Fall 2019

Exercise problem set #n?

1. **A walkthrough on vacuum entanglement**

   (a) Consider a massless scalar field in 2d Minkowski space. This field satisfies the classical wave equation:
   $$\left(-\partial_t^2 + \partial_x^2\right) \Phi(t, x) = 0 \ .$$
   Verify that the general solution to the wave equation can be written as:
   $$\Phi(t, x) = \int_0^\infty \frac{dk}{\sqrt{4\pi k}} \left[a_k e^{ik(x-t)} + a_k^\dagger e^{-ik(x-t)}\right] + (t \to -t,\ a_k \to a_{-k}) \ . \tag{1}$$
   To quantize this system, we declare a_k an annihilation operator and $a_k^\dagger$ a creation operator with commutation relation:
   $$\left[a_k, a_{k'}^\dagger\right] = \delta(k - k') \ , \qquad [a_k, a_{k'}] = \left[a_k^\dagger, a_{k'}^\dagger\right] = 0 \ .$$
   The vacuum is chosen to have no particles in it
   $$a_k|0\rangle_M = 0 \quad \forall k \ .$$

   (b) Now consider an observer in an accelerated frame obtained by the following coordinate transformation
   $$t = \frac{e^{a\xi}}{a} \sinh a\tau \ , \qquad x = \pm\frac{e^{a\xi}}{a} \cosh a\tau \ .$$
   These coordinates do not cover all of Minkowski space, but just a patch known as the left or right Rindler wedge, depending on the sign we choose in the x transformation above.

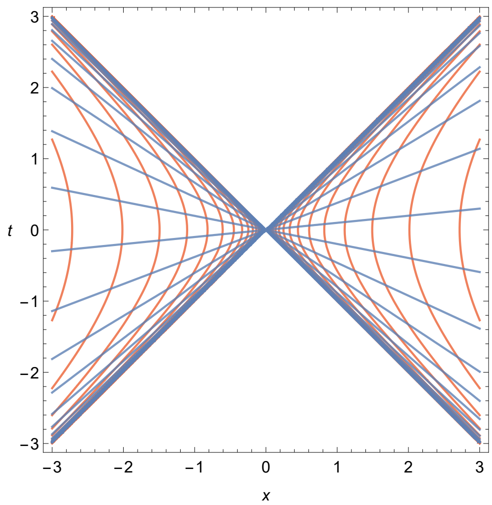

Figure 1: Lines of constant τ depicted in blue and lines of constant ξ depicted in red.

   Accelerating observers therefore do not have access to all of Minkowski space and see a horizon!
   Find the equations of motion for a massless scalar field in Rindler coordinates, solve them as in (1) and quantize the theory. What is the condition that defines the Rindler vacuum?

(c) In question 1b you should have found that the Rindler decomposition of fields are

$$\Phi^R = \int_0^\infty \frac{ds}{\sqrt{4\pi s}} \left[b_s^R \{a(x-t)\}^{is/a} + b_s^{R\dagger} \{a(x-t)\}^{-is/a} \right] + (t \to -t, b_s^R \to b_{-s}^R) \quad (2)$$

in the right rindler wedge and similarly

$$\Phi^L = \int_0^\infty \frac{ds}{\sqrt{4\pi s}} \left[b_s^{L\dagger} \{a(t-x)\}^{is/a} + b_s^{L} \{a(t-x)\}^{-is/a} \right] + (t \to -t, b_s^L \to b_{-s}^L) \quad (3)$$

in the left wedge. The Rindler observer's vacuum is defined as $b_s^{L/R}|0\rangle_R = 0$ for all s. Notice there seems to be an asymmetry between what we are calling a creation/annihilation operator in the right wedge and a creation/annihilation operator in the left wedge. This is because the time coordinate that moves forward in the right wedge does the opposite in the left wedge (stare at figure 1 if this point is confusing).

Express the Minkowski annihilation operator in terms of the left/right Rindler annihilaton and creation opertors. For this part of the problem, you need only consider the 'right moving' modes whose functional dependence is $t-x$. The 'left movers' will behave in exactly the same way.

Hint: write $\Phi = \theta(t-x)\Phi_L + \theta(x-t)\phi_R$ and use the orthonormality of the fourier modes. You should find

$$\left(b_s^R - e^{-\pi s/a} b_s^{L\dagger} \right) |0\rangle_M = \left(b_s^L - e^{-\pi s/a} b_s^{R\dagger} \right) |0\rangle_M = 0 \quad (4)$$

(d) Show that this implies that the Minkowski vacuum can be written as

$$|0\rangle_M = C \exp\left[- \int ds\, g(s)\, b_s^{L\dagger} b_s^{R\dagger} \right] |0\rangle_R$$

and find g. Hint: use the Baker-Campbell-Hausdorff formula

$$e^X Y e^{-X} = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \dots \quad (5)$$

2. **Hawking radiation generates too much entanglement**

In the previous problem you studied the entanglement properties of the vacuum state as viewed by an accelerating observer. By the equivalence principle, a similar structure of entanglement will hold outside of a black hole horizon. We want to explore what this means for Hawking evaporation.

(a) Consider matter in a pure state with wavefunction $|\psi_m\rangle$ collapsing to form a black hole. In order to understand the information problem, we will slice our geometry in such a way that our spatial slices are always weakly curved. This is done in order to avoid any recourse to quantum gravitational effects—after all Hawking's paradox can be entirely formulated within semiclassical gravity.

Based on the previous question, argue that the full quantum state should be given by

$$|\Psi\rangle = |\psi_m\rangle \otimes e^{g a_k^{L\dagger} a_k^{R\dagger}} |0_{\text{Schw}}\rangle$$

where k is a wavenumber on the scale of the black hole horizon, $|0_{\text{Schw}}\rangle$ is the vacuum of the Schwarzschild observer and L/R refer to modes inside/outside the horizon.

The details of this state aren't necessary for the argument so we will approximate as

$$|\Psi\rangle \approx |\psi_m\rangle \otimes \frac{1}{\sqrt{2}} \left(|0_L\rangle|0_R\rangle + |1_L\rangle|1_R\rangle\right)$$

Compute the reduced density matrix of the right subsystem and compute its entanglement entropy.

(b) The next step in the Hawking radiation process produces another pair of Hawking quanta, and our state is

$$|\Psi\rangle \approx |\psi_m\rangle \otimes \frac{1}{\sqrt{2}} \left(|0_{L_1}\rangle|0_{R_1}\rangle + |1_{L_1}\rangle|1_{R_1}\rangle\right) \otimes \frac{1}{\sqrt{2}} \left(|0_{L_2}\rangle|0_{R_2}\rangle + |1_{L_2}\rangle|1_{R_2}\rangle\right) \ .$$

What is the entanglement entropy of the right subsystem now? And after N such steps?

(c) Let us assume that the black hole is completely evaporated after N such steps. Argue why the result you derived is inconsistent with unitarity.

(d) *Extra credit: Propose a resolution.*