De Sitter QFT Lectures ($\hbar = c = 1$)

Further Reading: 0110007 (Les houches lectures) | Bousso, Strominger, Maloney 0112218
Birrell & Davies § 5.4
1205.3855 De Sitter Musings

Geometry: dS is the unique maximally symmetric space time whose metric satisfies: Constant curvature ✓

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \quad \text{for } \Lambda > 0$$

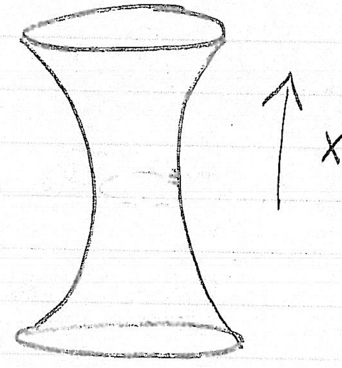
We can define a de Sitter length: $\boxed{\ell^2 = \frac{(d-2)(d-1)}{2\Lambda}}$

Being maximally symmetric, dS has $\frac{d(d+1)}{2}$ isometries

Flat space is also MS: d translations, $(d-1)$ boosts, $\frac{(d-1)(d-2)}{2}$ rotations

It can be realized as the hypersurface satisfying

$$-X^{0\,2} + X^{1\,2} + \ldots + X^{d\,2} = \ell^2$$



The metric of dS_d can be induced from the metric on $\mathcal{M}_{d+1}$

$$ds^2 = -dX^{0\,2} + dX^{1\,2} + \ldots + dX^{d\,2}$$

The isometries of dS_d are inherited from the isometries of $\mathcal{M}_{d+1}$

↳ Exercise: which are they?

→ There are therefore no globally timelike isometries of dS

So dS_d is $\mathbb{R} \times S^{d-1}$, and the spheres are (expanding / contracting)

<u>Coordinates</u> Since de Sitter is $\mathbb{R} \times$ sphere, we need a good way to parametrize coordinates on the sphere:

define
$$\omega^1 = \cos\theta_1$$
$$\omega^2 = \sin\theta_1 \cos\theta_2$$
$$\omega^{d-1} = \sin\theta_1 \dots \sin\theta_{d-2} \cos\theta_{d-1}$$
$$\omega^d = \sin\theta_1 \dots \sin\theta_{d-2} \sin\theta_{d-1}$$
$$\theta_i \in [0, \pi] \quad 1 \le i \le d-2 \quad , \theta_{d-1} \in [0, 2\pi]$$

$$\boxed{d\Omega_{d-1}^2 = \sum_{i=1}^{d} (d\omega^i)^2}$$

<u>Global coords:</u>
$$X^0 = l \sinh\tau \qquad \tau \in [-\infty, \infty]$$
$$X^i = l \cosh\tau \, \omega^i$$

$$ds^2 = l^2(-d\tau^2 + \cosh^2\tau \, d\Omega_{d-1}^2)$$

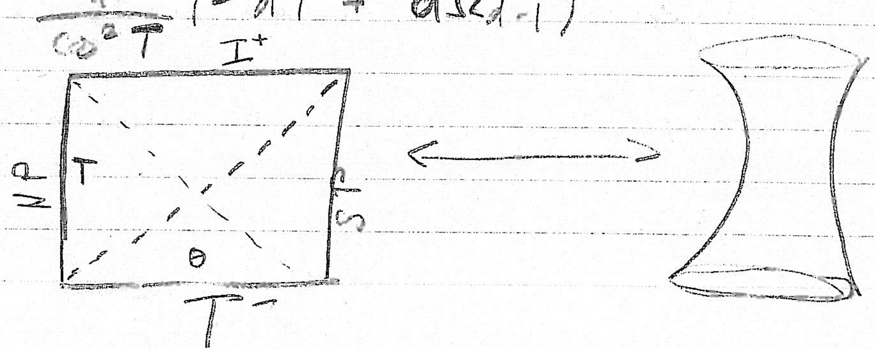
→ These coordinates cover all of dS_d

<u>Penrose diagram & conf coords</u>

Let $\cosh\tau = \frac{1}{\cos T}$, $T \in [-\pi/2, \pi/2]$

$$ds^2 = \frac{l^2}{\cos^2 T}(-dT^2 + d\Omega_{d-1}^2)$$

<u>each point is S^{d-1}</u>

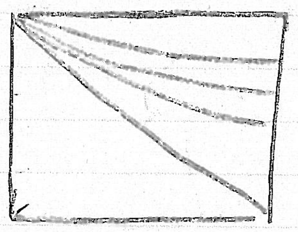


FRW / Planar coords

$$X^0 = \ell \sinh t/\ell - \frac{1}{2\ell}\vec{x}\cdot\vec{x}\, e^{+t/\ell}$$
$$X^i = x^i e^{+t/\ell}$$
$$X^d = \ell \cosh t/\ell - \frac{1}{2\ell}\vec{x}\cdot\vec{x}\, e^{t/\ell}$$

$$ds^2 = -dt^2 + e^{2t/\ell} d\vec{x}^2 = \frac{\ell^2}{\eta^2}(-d\eta^2 + d\vec{x}^2), \quad \eta = -\ell e^{-t/\ell}$$

$t \in [-\infty, \infty]$, $\eta \in [-\infty, 0^-]$ $\rightarrow$ covers half of dS

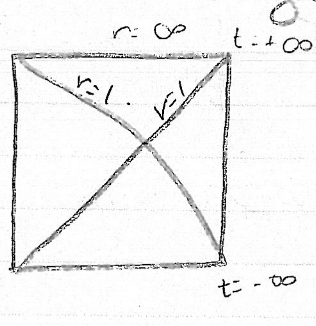


Static Patch coordinates

$$X^0 = \sqrt{\ell^2 - r^2}\sinh(t/\ell)$$
$$X^a = r\,\omega^a \qquad a = 1, \ldots, d-1, \quad \omega^a \text{ parametrize on } S^{d-}$$
$$X^d = \sqrt{\ell^2 - r^2}\cosh(t/\ell)$$

$$ds^2 = -(1 - r^2/\ell^2)dt^2 + \frac{dr^2}{(1 - r^2/\ell^2)} + r^2 d\Omega^2_{d-2}$$

∂_t is a killing symmetry in this coordinate system but it is not globally timelike $\longrightarrow$ horizon



These coords cover a quarter of dS, but are natural coordinates for geodesic observers in dS.

Geodesics Let us Euclideanize minkowski

Then our equation for the embedding of dS_d is

$$\eta_{ij} X^i X^j = \ell^2 \rightarrow \delta_{ij} X^i X^j = \ell^2 \qquad S^{d+1} \text{ of radius } \ell$$

The geodesic distance^2 between two points X & X' on a sphere of radius ℓ: $D^2 = \ell^2 \theta^2$, where $\theta \equiv$ angle subtend

$$\delta_{ij} X^i X^{j'} = \ell^2 \cos\theta, \quad \therefore \quad \theta = \cos^{-1} \frac{\delta_{ij} X^i X^{j'}}{\ell^2}$$

$$\text{\&} \quad D^2 = \ell^2 \left(\cos^{-1} \frac{\delta_{ij} X^i X^{j'}}{\ell^2} \right)^2$$

$\rightarrow$ To go back to Lorentzian dS: $\boxed{D^2 = \ell^2 \left(\cos^{-1} \frac{\eta_{ij} X^i X^{j'}}{\ell^2} \right)^2}$

$$P = \eta_{ij} X^i X^{j'} / \ell^2 = \cos \sqrt{\frac{D^2}{\ell^2}}$$

Timelike :	$D^2 < 0$	$P > 1$
Spacelike :	$D^2 > 0$	$-1 < P < 1$
Null :	$D^2 \equiv 0$	$P = 1$
Antipodal :	$D^2 = \ell^2 \pi^2$	$P = -1$

Little known fact : $P < -1$ describes points not separated by geodesic paths. D^2 complex

Said in another way, all space like geodesic curves lie on a circle of radius $2\pi\ell$. (1993 paper by Jurgen Schmidt)

dS is not space-like geodesic complete.

Quantum Fields in dS

$$S = \frac{1}{2}\int d^d x \sqrt{-g}\left\{-g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - m^2\phi^2\right\}$$

$$= \frac{1}{2}\int d^d x\, \phi\left\{\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right) - \sqrt{-g}\,m^2\phi\right\}$$

EOM: $\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\phi\right) = +m^2\phi$

Any quantum field can be decomposed into modes φ_n (n represents the quantum numbers) which satisfy the EOM

$$\varphi(x) = \sum_n \left[a_n u_n(x) + a_n^\dagger u_n^*(x)\right]$$

The individual modes form a complete basis orthonormal under the norm:

$$(u_1, u_2) = -i\int_\Sigma d\Sigma\,\left\{u_1\partial_\mu u_2^* - (\partial_\mu u_1)u_2^*\right\}n^\mu$$

where Σ is a complete Cauchy slice & n^μ is the unit normal.

When quantizing fields we wish to impose the canonical commutation relations:

$$[\varphi(t,\vec{x}),\ \varphi(t,\vec{x}')] = 0$$
$$[\pi(t,\vec{x}),\ \pi(t,\vec{x}')] = 0$$
$$[\varphi(t,\vec{x}),\ \pi(t,\vec{x}')] = \frac{i\,\delta^{d-1}(\vec{x}-\vec{x}')}{\sqrt{g_\Sigma}}$$

Where $\pi(t,x) = \frac{\partial\mathcal{L}}{\partial(\partial_t\varphi)}$

This implies: $[a_n, a_m] = [a^+_n, a^+_m] = 0$, $[a_n, a^+_m] = \delta_{nm}$

The vacuum, or "no particle" condition is defined by:

$$a_n|\Omega\rangle = 0 \quad \text{for all } n$$

In Minkowski we use the modes: $\sim e^{-i(\omega t - \vec{k}\cdot\vec{x})}$; $\omega > 0$, $\omega^2 = k^2 + m^2$
to annihilate the vacuum. These modes are eigenfunctions of the globally timelike Killing vector ∂_t

In dS, there is no globally timelike K-vector, so our choice of modes/vacuum is a bit arbitrary.

Let us consider the Wightman two point function:

$$G_\Omega = \langle\Omega|\phi(x)\,\phi(y)|\Omega\rangle = \sum_n u_n(x)\,u_n^*(y)$$

necessary discuss ret/adv etc...

→ Since the choice of modes characterizes the Vacuum then there is a 1 to 1 correspondence between $|\Omega\rangle$ & G_Ω

G_Ω satisfies the scalar wave equation, & by dS symmetry should only depend on P

$$(1-P^2)\,G''_\Omega(P) - d\,P\,G'_\Omega(P) - m^2 l^2 G_\Omega(P) = 0$$

Two solutions: $G_\Omega = A\,{}_2F_1\left(h_+, h_-, \frac{d}{2}, \frac{1+P}{2}\right) + B\,{}_2F_1\left(h_+, h_-, \frac{d}{2}, \frac{1-P}{2}\right)$

D: $A = \dfrac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^{d/2}\, l^{d-2}\,\Gamma(\frac{d}{2})}$

$$h_\pm = \frac{1}{2}\left\{(d-1) \pm \sqrt{(d-1)^2 - 4m^2 l^2}\right\}$$

A) has a singularity on the lightcone $(P=1)$ and decays for large separation

B) has a singularity at $P=-1$ for points antipodally separated, this singularity appears unphysical, but can't be detected. Decays for large separation

Minkowski inside Wightman function satisfies

$$+2\sigma F''(\sigma) + d\, F'(\sigma) - m^2 F(\sigma) = 0 \qquad \sigma = \frac{1}{2}(x-x')^2 = D^2$$

$$F(\sigma) = \begin{cases} \dfrac{B\, I_{\frac{d}{2}-1}(\sqrt{2m\sigma}) + A\, K_{\frac{d}{2}-1}(\sqrt{2m\sigma})}{\sigma^{\frac{d-2}{4}}} & d \text{ even} \\[2ex] \dfrac{B\, e^{\sqrt{2m\sigma}} + A\, e^{-\sqrt{2m\sigma}}}{\sigma^{\frac{d-2}{2}}} \quad (x^n \ldots) & d \text{ odd} \end{cases}$$

⇝ Can always rule one solution out because it does not decay at $\sigma \to \infty$

Case for $B=0$ the vacuum corresponding to $B=0$ is known as the Bunch-Davies or Euclidean vacuum

→ Gives the proper Minkowski singularity at short distances $\sim (D^2)^{1-d/2}$ ✓

Which modes give the BD vacuum?

Minkowski modes: $\dfrac{1}{\sqrt{(2\pi)^{d-1}\, 2\omega_k}}\, e^{i(\vec{k}\cdot\vec{x} - \omega_k t)}$ $t \to -\infty$

Planar modes: $\varphi(\eta) = \dfrac{(-\eta)^{\frac{d-1}{2}}}{2\sqrt{2(2\pi\ell)^{d-2}}}\, H_1^{\nu}(-k\eta)$, $\dfrac{(-\eta)^{\frac{d-1}{2}}}{2\sqrt{2(2\pi\ell)^{d-2}}}\, H_2^{\nu}(-k\eta)$

$$\nu = \frac{1}{2}\sqrt{(d-1)^2 - 4m^2\ell^2}$$

For large k Hankel $H_2 \sim \dfrac{1}{\sqrt{(2\pi)^{d-1}\, 2k}}\, e^{i(\vec{k}\cdot\vec{x} - k\eta)}$

↳ These define the BD vacuum!

The continuous family of vacua are called the α vacua of dS

There are many specific vacua of interest $|in\rangle, |out\rangle$ that I will discuss in Lecture 2.

<u>Temperature of dS</u> (Birell & Davies, Strominger Les Houches)

We just spent some time discussing that there are many inequivalent vacua in dS, each of which do not agree on the Fock space.

However, we may ask what a geodesic detector (Unruh) might uncover.

In flat space, these inertial detectors detect no particles.

Can model detector by coupling a scalar field along the worldline of the detector $x^\mu(\tau)$ to an operator which acts on the internal states of the detector

(τ is the proper time of the detector)

The coupling looks like $\underbrace{g}_{\text{coupling constant}}\int_{-\infty}^{\infty} d\tau\, m(\tau)\, \phi(x(\tau))$

Let H be the Hamiltonian for the detector with energy eigenstates $|E_i\rangle$

→ I can choose my detector to have these properties!

Define $m_{ij} = \langle E_i | m(0) | E_j\rangle$

$\rightarrow$ any state

We want to calculate $|0\rangle|E_i\rangle \rightarrow |\Psi\rangle|E_f\rangle$

The transition amplitude: $\langle E_f|\langle\Psi|\, e^{g\int m(\tau)\phi[x(\tau)]d\tau}|0\rangle|E_i\rangle$

$$\sim g\int_{-\infty}^{\infty} d\tau\, \langle E_f|\langle\Psi|\, m(\tau)\phi[x(\tau)]|0\rangle|E_i\rangle\,,\quad m(\tau) = e^{iH\tau}m(0)e^{-iH\tau}$$

$$= g\, m_{ji}\int_{-\infty}^{\infty} d\tau\; e^{i(E_f-E_i)\tau}\langle\Psi|\phi[x(\tau)]|0\rangle$$

Since we are interested in $P(E_i\rightarrow E_f)$, we square the amplitude & sum over the intermediate state $|\Psi\rangle$

$$P(E_i\rightarrow E_f) = g^2|m_{ij}|^2\int d\tau\, d\tau'\; e^{-i(E_f-E_i)(\tau'-\tau)}\, \underbrace{G}_{\rightarrow\ \text{Wightman}}(P(x(\tau'),x(\tau)))$$

Now let us assume the trajectory is a geodesic in static coordinates at the origin.

$$P(x(\tau')\, x(\tau)) = \cosh\left(\frac{\tau'-\tau}{L}\right),$$

$$y = \frac{(\tau'+\tau)}{2},\quad \Delta\tau = \tau'-\tau\,,\quad \tau' = y+\frac{\Delta\tau}{2}\,,\quad \tau = y-\frac{\Delta\tau}{2}$$

$$\begin{bmatrix} \partial\tau'/\partial y & \partial\tau'/\partial\Delta\tau \\ \partial\tau/\partial y & \partial\tau/\partial\Delta\tau \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} \\ 1 & -1/2 \end{bmatrix} \Rightarrow -1$$

$$\frac{P(E_i\rightarrow E_f)}{T\rightsquigarrow\text{total time}} = -g^2|m_{ij}|^2\int_{-\infty}^{\infty} d\Delta\tau\; e^{-i(E_f-E_i)\Delta\tau}\; G\left(\cosh\Delta\tau/\ell\right)$$

$m^2_{conf} = d(d-2)/4\ell^2$, For simplicity let us assume $d = 4 \Rightarrow m = m^2_{conf}$

$$\frac{P(E_i\rightarrow E_f)}{T} = +\frac{g^2|m_{ij}|^2}{16\pi^2\ell^2}\int_{-\infty}^{\infty} d\Delta\tau\left\{\frac{e^{-iE\Delta\tau}}{\sinh^2\left(\frac{\Delta\tau}{2\ell}\right)} = -\frac{e^{-iE\Delta\tau}}{\sin^2\left(\frac{i\Delta\tau}{2\ell}\right)}\right\}$$

$$\operatorname{cosec}^2\pi x = \frac{1}{\pi^2}\sum_n (x-n)^{-2}$$

$$G_{m^2 conf} = \frac{\Gamma(d/2-1)}{(4\pi)^{d/2}\ell^{d-2}}\times\frac{1}{(-\sinh^2(\Delta\tau/2\ell))^{d-2}}$$

$$\therefore \quad \frac{P(E_i \to E_f)}{T} = \frac{g^2 |m_{ij}|^2}{16\pi^4 \ell^2} \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} d\Delta\tau \, \frac{e^{-iE\Delta\tau}}{\left(\frac{i\Delta\tau}{2\pi\ell} - n + \epsilon\right)^2}$$

~~not important~~

Close contour in bottom half-plane, $n > 0$

$$\int_{-\infty}^{\infty} d\Delta\tau \, \frac{e^{-iE\Delta\tau}}{\left(\frac{i\Delta\tau}{2\pi\ell} - n\right)^2} = -4\pi^2\ell^2 \int_{-\infty}^{\infty} d\Delta\tau \, \frac{e^{-iE\Delta\tau}}{(\Delta\tau + i2\pi n\ell)^2}$$

$$= -4\pi^2\ell^2 \int_{-\infty}^{\infty} d\Delta\tau \; e^{-2\pi n\ell E} \left\{ \frac{1}{(\Delta\tau + i2\pi n\ell)^2} - \frac{iE}{(\Delta\tau + i2\pi n\ell)} + \dots \right.$$

$$= -8\pi^3 \ell^2 E \, e^{-2\pi n\ell E}$$

$$\therefore \quad \frac{P(E_i \to E_f)}{T} = g^2 |m_{ij}|^2 \left(\frac{E}{2\pi}\right) \frac{1}{e^{2\pi\ell E} - 1}$$

$\rightsquigarrow$ This is a thermal distribution

$(E = E_f - E_i)$ with $\boxed{T = \frac{1}{2\pi\ell}}$

↳ Pathria ch. 7

<u>More slick argument (Andy)</u> (not assuming m^2 conf, $d = 4$)

Assume: $\frac{P(E_i \to E_f)}{T} = \frac{P(E_f \to E_i)}{T} e^{-\beta(E_f - E_i)}$ (1)

& that the energy levels of the detector are thermally populated: $N_i = N e^{-\beta E_i}$ (2)

$\therefore \; R(E_i \to E_f) = N e^{-\beta E_i} \frac{P(E_i \to E_f)}{T} = R(E_f \to E_i)$ (3)

$\rightsquigarrow$ So if the two conditions hold, then N_i does not change over time & the system is thermal

(3) → equilibrium, so given (1) & (3), (2) must hold.

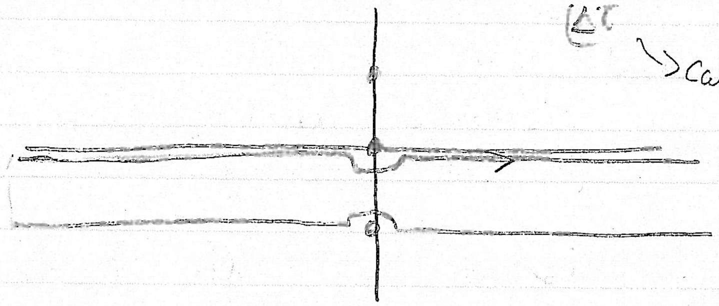
must prove (1)

singular pt

Consider $\frac{P(E_i \to E_f)}{T} \propto \int_{-\infty}^{\infty} d\Delta\tau \, e^{-i(E_f - E_i)\Delta\tau} \, G(\cosh(\Delta\tau / L))$

& Poles when $\Delta\tau = 2\pi i n \ell$ The integral around this contour is zero

$\Delta\tau$ called Matsubara frequencies



$$\therefore 0 = \int_{-\infty}^{\infty} d\Delta\tau \, e^{(E_f - E_i)\Delta\tau} G(\cosh(\Delta\tau / L)) + \int_{+\infty - i2\pi\ell}^{-\infty - i2\pi\ell} d\Delta\tau \, e^{-(E_f - E_i)\Delta\tau} G(\cosh(\Delta\tau / L))$$

$$\rightarrow \frac{P(E_i \to E_f)}{T} = e^{-\beta(E_f - E_i)} \frac{P(E_f \to E_i)}{T}$$

<u>Open problem</u>

I have told you many interesting properties about dS, but I haven't told you what we are interested in when we study quantum fields in de Sitter.

What are the observables? In flat space we know that it is the S-matrix. This is not observable in dS, because of the horizon & because of the lack of an asymptotic region.

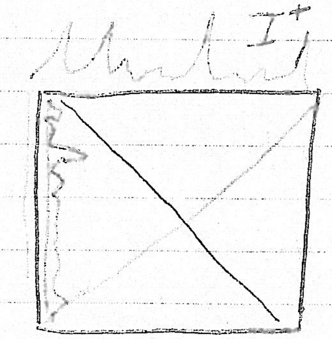
→ Exercise: Think about what good observables are in dS.

In a gen vacuum: $\frac{P(\ldots)}{P(E_f \to E_i)} = e \ldots$ $\left|\frac{1+e^{\ldots+\pi\alpha}}{1+e^{\alpha-\pi\beta E}}\right|$

Further Reading: Witten 0106109, Maldacena 0210603, Anninos, Hartman, Hofman ... 1109.4942

Lecture 2 (speculative)

What are the proper observables in dS? There are two perspectives to take on this. One is that of an observer sitting at the south pole who does not have access except to a small region around her worldline, and the other is one of an observer "metaobserver" who has access to all field correlations @ $\mathcal{I}^+$



- We are currently the meta observers of the previous dS era of inflation (CMB) as well as being worldline observers of the current dS era

Worldline perspective

$R(L, r) : \ldots$

If we are considering the physics of the worldline then the natural perspective is that of a quantum mechanics

Let us consider a non-interacting a non-interacting scalar field in the static patch

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Omega_{d-2}$$

The natural modes to consider are $\phi(t, r, \Omega) = \varphi(r) e^{-i\omega t} Y_L(\Omega)$

Solve:
$$-\frac{1}{r^{d-2}}\frac{d}{dr}\left(f r^{d-2}\frac{d\varphi}{dr}\right) + \left(m^2 + \frac{L(L+d-3)}{r^2} - \frac{\omega^2}{f}\right)\varphi(r) = 0$$

with boundary conditions such that the waves of the ϕ field are only ingoing at the horizon $r = l$

Two solutions: $\varphi = B\varphi_n + A\varphi_{nn}$

$$\varphi_n = \left(1-\frac{r^2}{\ell^2}\right)^{-i\omega\ell/2}\left(\frac{r}{\ell}\right)^{L} {}_2F_1\left\{\frac{1}{2}(h_-+a), \frac{1}{2}(h_++a), \frac{d-1}{2}+L, \frac{r^2}{\ell^2}\right\}$$

$$\varphi_{nn} = \left(1-\frac{r^2}{\ell^2}\right)^{-i\omega\ell/2}\left(\frac{r}{\ell}\right)^{3-d-L} {}_2F_1\left\{\frac{1}{2}(h_-+b), \frac{1}{2}(h_++b), \frac{5-d}{2}-L, \frac{r^2}{\ell^2}\right\}$$

$$a = L - i\omega\ell, \quad b = 3-d-L-i\omega\ell$$

$$L \Longleftrightarrow 3-d-L$$

We want modes to only go into the horizon

e.g. need modes to go like $\left(1-\frac{r^2}{\ell^2}\right)^{-i\omega\ell/2}$ near $r=\ell$

This gives $$\frac{B}{A} = -\frac{\Gamma\left(\frac{5-d}{2}-L\right)}{\Gamma\left(\frac{d-1}{2}+L\right)} P_+(\omega)P_-(\omega)$$

$$P_\pm(\omega) = \frac{\Gamma\left[\frac{1}{2}(L-i\omega\ell+h_\pm)\right]}{\Gamma\left[\frac{1}{2}(3-d-L-i\omega\ell+h_\pm)\right]}$$

This ratio has poles when $\boxed{i\omega\ell = 2n+L+h_\pm \quad n\geq 0}$ which are known as the quasinormal modes of dS, e.g. these are the modes which are normalizable in the interior as this linear combination suppresses the behavior of φ_{nn}.

Let us compute the Fourier transform of $\frac{B}{A}$ for $m^2 = \frac{d(d-2)}{4\ell^2}$

$$F_t\left[\frac{B}{A}\right] \propto \Theta(t)\left(\sinh\left(\frac{t}{2\ell}\right)\right)^{2-d-2L}$$

compare with dS invariant Wightman function for points separated along worldline:

$$G \propto \left(\sinh \Delta t/_{2\ell}\right)^{2-d}$$

So $\frac{B}{A}$ resemble the frequency space retarded Green's functions along a worldline

We will now show that the retarded propagator for the conformally coupled mass, is precisely one for a particular type of operator in a conformal quantum mechanics. I will not discuss the general case, which is interesting and more complicated, but refer you to (Anninos, Hartnoll, Hofman).

<u>Conformal quantum mechanics</u> has D, H, K which satisfy:

$$[D,H] = -iH, \quad [D,K] = +iK, \quad [H,K] = 2iD$$

Aside

take a state $H|E\rangle = E|E\rangle$, then $\quad HD = DH + [H,D]$

$H e^{-i\alpha D}|E\rangle = \quad \ldots \quad + i\alpha[D,H] = + \alpha(iH)$

use $Ae^{B} = e^{B}\left(A - [B,A] + \frac{1}{2!}[B,[B,A]] + \quad\right)$

$H e^{-i\alpha D}|E\rangle = e^{-i\alpha D}\left(H + \alpha H + \frac{\alpha^2}{2!} H + \ldots\right)|E\rangle$

$= e^{\alpha} E\, e^{-i\alpha D}|E\rangle$, spectrum is continuous and energy eigenstates are not normalizable

<u>Let us look at the quasinormal modes φ_n in the conf case</u>

$$\varphi_n = \left(\frac{r}{\ell}\right)^{L} f^{-i\omega_n \ell/2}\, {}_2F_1\left(\frac{(1-n)}{2}, -\frac{n}{2}, \frac{d-1}{2}+L, \frac{r^2}{\ell^2}\right) e^{-i\omega_n t}$$

$$\omega_n \ell = -i\left(n + \frac{d}{2} + L - 1\right)$$

Let: $H_{\pm} = i\,\frac{e^{\pm t/\ell}}{\sqrt{f}}\left(\pm r f \frac{\partial}{\partial r} + \ell\, \partial_t \pm f\right)$

$$H_+ \phi_m = -i\left\{m + \frac{(d-4)}{2}\right\}\phi_{m-1}, \quad H_- \phi_m = +i\left\{m + \frac{(d-4)}{2} - 2i\omega_m \ell\right\}\phi_{m+1}$$
$$-i\left\{m + 2\ell + \frac{d}{2}\right\}\phi_{m+1}$$

$\frac{d\,?}{d t_0}$ ...

$d_{dio} = d_T - 1$

isometr(

Easy to show that $[H_+, H_-] = -2iH_0$, with $H_0 = i\ell\partial_t$
& $[H_\pm, H_0] = \mp i H_\pm$

These operators are related to the regular $SL(2,\mathbb{R})$ ones by

$$D = \frac{i}{2}(H_- - H_+), \quad K = -H_0 - \frac{1}{2}(H_+ + H_-), \quad H = H_0 - \frac{1}{2}(H_+ + H_-)$$

Notice that the isometry $H_0 = (H-K)/2$

And that these $H_\pm$ are not isometries but conformal isometries

Casimir: $\left\{-D^2 + \frac{1}{2}(HK + KH)\right\}\phi = L(L+1)\phi$

Doesn't match dio's result

$L(L+1) = \Delta(\Delta-1)$, $\Delta = L+1$, So these are operators with definite conformal weight

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<u>Toy model</u> $H = i\ell \frac{d}{d\tau}$, $D = i\tau\frac{d}{d\tau}$, $K = \frac{i}{\ell}\tau^2\frac{d}{d\tau}$

$$H_0 = i\ell\partial_t = \frac{1}{2}(H-K) \longrightarrow \tau = \ell \tanh(t/2\ell)$$

We will see that this simple model captures the physics of the QNMs

Define $L_\pm = iD - H_0$ & $R = \frac{1}{2}(H+K)$

These satisfy: $[R, L_\pm] = \pm L_\pm$, $[L_-, L_+] = 2R$

The primary operators satisfy $R|0\rangle = r_0|0\rangle$, $L_-|0\rangle = 0$

If we define a new state: $|\tau\rangle = N(\tau)e^{-\omega(\tau)L_+}|0\rangle$

with $N(\tau) = \sqrt{\Gamma(2r_0)}\left(\frac{\omega(\tau)+1}{2}\right)^{2r_0}$, $\omega(\tau) = \frac{\ell + i\tau}{\ell - i\tau}$

Then we can show that $\langle \tau' | \tau \rangle = \dfrac{\Gamma(2r_0)\, \ell^{2r_0}}{(2i(\tau'-\tau))^{2r_0}}$

$\rightsquigarrow$ not quite what we want. Define

$$|t\rangle_{dS} = \left(\left(\frac{\tau}{L}\right)^2 - 1\right)^r |\tau\rangle \quad , \quad \langle t' | t \rangle \propto \left(\frac{1}{\sinh \frac{1}{2}(t'-t)}\right)^{2r}$$

$\rightsquigarrow$ Are these eigenvalues of H_0? <u>Should check</u>

In the general case, there are two $SL(2,\mathbb{R})$s that must be related by level matching etc. but similar structure exists.

Open question: is the physics of the dS observer captured by a conformal quantum mechanics?

<u>Part II</u> Metaobservers & the wavefunction.

Consider the path integral in quantum mechanics:

$$\int_{\substack{q(t_2)=q_2 \\ q(t_1)=q_1}} \mathcal{D}q\, e^{iS[q]} = \langle q_2; t_2 | q_1; t_1 \rangle = \langle q_2 | e^{-iH(t_2-t_1)} | q_1 \rangle$$

Insert a complete set of energy eigenstates $\sum_E |E\rangle\langle E| =$

$$= \sum_E \psi_E(q_2)\, \psi_E^*(q_1)\, e^{-iE(t_2-t_1)} \quad . \text{ Let } t = +i\tau$$

$$= \sum_E \psi_E(q_2)\, \psi_E^*(q_1)\, e^{+E\tau_2}\, e^{-E\tau_1}$$

Take $\tau_1 \to +\infty$ $\quad q_1 = 0$ $\quad$ selects out the ground state
$\tau_2 \to \tau_c$ $\quad q_2 = x$

$$\psi_g(x) = \int_{\substack{q(\tau_c)=x \\ q(\infty)=0}} \mathcal{D}q\, e^{-S_E[q]} \quad \text{with } S_E = \int_{\tau_c}^{\infty} d\tau \left(\frac{\dot{q}^2}{2} + V(q)\right)$$

We can even break up the fields into $q = q_{cl} + \delta q$: $\Psi_g(x,\tau_c) = e^{-S[q_{cl}]} \int D\delta q \, e^{-S_E[\delta q]}$

If $V(q)$ is time dependent: $H\Psi = i\partial_t \Psi$

What does this have to do with dS?

Consider dS in Poincaré slicing $ds^2 = \frac{l^2}{\eta^2}(-d\eta^2 + d\vec{x}^2)$, $\eta \in [-\infty, 0]$

with a scalar field with Lagrangian:

$$S = -\int d^d x \sqrt{-g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right\}$$

& Let $\phi(\eta, \vec{x}) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}} e^{i\vec{k}\cdot\vec{x}} \phi_{\vec{k}}(\eta)$

The Bunch Davies Partition function $Z(\varphi_{\vec{k}}, \eta_c) = \int D[\phi_{\vec{k}}] e^{iS[\phi_{\vec{k}}]}$

is given by the path integral where we demand

$\phi_{\vec{k}} \sim e^{ik\eta}$ as $k\eta \to -\infty$ and $\phi_{\vec{k}}(\eta_c) = \varphi_{\vec{k}}$

Let us now take $\eta = iz$ & $l = iL_A$

The boundary conditions continue to $\phi_{\vec{k}} \sim 0$ as $z \to +\infty$
$\phi_{\vec{k}}(z_c) = \varphi_{\vec{k}}$

& The new metric is $ds^2 = \frac{L_A^2}{z^2}(dz^2 + d\vec{x}^2) \to$ EAdS

This seems to suggest:

$$\Psi_{BD}[\varphi_{\vec{k}}, \eta_c] = Z_{EAdS}[\varphi_{\vec{k}}, z_c = -i\eta_c] \Big|^{L_A = -l}_{\substack{\phi_{\vec{k}}(z_c) = \varphi_{\vec{k}} \\ \phi_{\vec{k}}(\infty) = 0}}$$

At least at the level of perturbative quantum fields in dS

Let's see how this works in practice

Wish to solve the boundary value problem:

$$\left[z^2\partial_z^2 - (d-2)z\partial_z - (k^2z^2 + m^2L_A^2)\right]\phi_k(z) = 0 \qquad \phi_k(z_c) = \varphi_k$$

The Damped solutions are given by: $\phi_k(z) = K(z,k)\varphi_k = \dfrac{z^{(d-1)/2}K_\nu(kz)}{z_c^{(d-1)/2}K_\nu(kz_c)}$

$\nu = \frac{1}{2}\sqrt{(d-1)^2 + 4m^2L_A^2}$

In the classical approximation $Z_{AdS} = e^{-S_{cl}(\varphi_k)}$

$$S_{cl}[\varphi_k] = -\frac{1}{2}\int \frac{dk}{(2\pi)^{d-1}}\left(\frac{L_A}{z}\right)^{d-2} K(z,k)\partial_z K(z,k)\,\varphi_k\varphi_{-k}\Big|_{z=z_c}$$

For a massless scalar in $d=4$, $K(z,k) = \dfrac{(1+kz)e^{-kz}}{(1+kz_c)e^{-kz_c}}$

$$S_{cl} = \frac{1}{2}\int \frac{d\vec{k}}{(2\pi)^3}\left(\frac{L_A}{z_c}\right)^2 \frac{k^2 z_c}{1+kz_c}\varphi_k\varphi_{-k}$$

Taking $z_c \to 0$ as is usually done in AdS/CFT

$$S_{cl} = \frac{1}{2}\int \frac{d\vec{k}}{(2\pi)^3}\left(\frac{L_A}{z_c}\right)^2\left\{k^2 z_c - k^3 z_c^2 + \dots\right\}\varphi_k\varphi_{-k}$$

When we continue back to dS, $z_c \to -i\eta_c$, $L_A \to i\ell$

$$\log \Psi_{BD} = \frac{\ell^2}{2}\int \frac{dk}{(2\pi)^3}\left(\frac{ik^2}{\eta_c} - k^3\right)\varphi_k\varphi_{-k}$$

Phase, in AdS, can add counterterm

Using Ψ_{BD} to compute $\langle\varphi_k\varphi_{-k}\rangle = \dfrac{1}{2\ell^2k^3}$ → Scale invariant spectrum.

⇝ Observation originally made by Maldacena.

Florence lao universality , Percy Deift "Univ in phys & Math"

Forrester: "Random Matrices & log gases" Ch. 1

Mehta: Random //

Can now understand no boundary wavefunction.

De Sitter Entropy

Put black hole in dS_d

$$ds^2 = -\left(1 - \frac{r_0}{r^{d-3}} - \frac{r^2}{\ell^2}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_0}{r^{d-3}} - \frac{r^2}{\ell^2}\right)} + r^2 d\Omega_{d-2} \qquad d > 2$$

$$r_0 = \frac{8MG}{(d-2)\pi^{(\frac{d-3}{2})}} \Gamma\left(\frac{d-1}{2}\right), \qquad d=3: \ 8MG \quad d=4: \ 2MG$$

M corresponds to the Schwarzschild Mass.

Need to solve $d-1$ order equation to find horizons.

In $d=4$: $r^3 - \ell^2 r + 2MG\ell^2 = 0$, for 3 real solutions, need $\ell^2 > (3\sqrt{3}MG)^2$

Solutions $r_k = \frac{2\ell}{\sqrt{3}} \cos\left(\frac{1}{3}\left\{2\pi k - \arccos\left(-\frac{3\sqrt{3}GM}{\ell}\right)\right\}\right)$

$r_0 \geq r_1 \geq 0 > r_2$

$$T_H \Rightarrow \frac{(r-r_0)(r-r_1)(r-r_2)}{r\ell^2} dt^2 + \frac{r^2\ell^2 \, dr^2}{(r-r_0)(r-r_1)(r-r_2)}$$

$$\boxed{T_H = \frac{(r_0-r_1)(r_0-r_2)}{4\pi\ell^2 r_0}}, \quad \text{reduces to } \frac{1}{2\pi\ell} \text{ for } M \to 0$$

$$T_{BH} = \frac{(r_0-r_1)(r_1-r_2)}{4\pi\ell^2 r_1}$$

Now in QG, we know that $S \propto A_H = 4\pi r_0^2$ & that it should satisfy a first law of thermodynamics

$$\frac{dS}{dM} = \frac{1}{T}, \quad \text{Take } S = cA, \text{ then we find}$$

$$c = -\frac{1}{4G} \ !$$

For the black hole, we find $\frac{dS_{BH}}{dM} = \frac{1}{T_{BH}}$ $\quad S_{BH} = \frac{A_{BH}}{4G}$

This makes sense, adding M decreases the cosmological horizon, so it is sensible to get $\boxed{\frac{dS_c}{d(-M)} = \frac{1}{T}}$

We don't really know how to think of the dS entropy, what exactly is it counting?

- Is it the free energy of the world like QM?
- Is it an entanglement entropy from tracing out the state outside the horizon?
- Is it captured by the central charge of some CFT via a Cardy-like formula?

$\rightarrow$ Homework $\longrightarrow$ What the F is it?